Worcester County Mathematics League

Varsity Meet 2 - December 1, 2021

COACHES' COPY
ROUNDS, ANSWERS, AND SOLUTIONS

Worcester County Mathematics League
Varsity Meet 2 - December 1, 2021
Round 1 - Fractions, Decimals, and Percents

All answers must be in simplest exact form.

1. A decrease of 30% followed by an increase of 10% is equivalent to an increase of 40% followed by a decrease of x%. Find x.

2. Convert the repeating decimal $0.2\overline{17}$ to a fraction $\frac{a}{b}$ in lowest terms.

3. The value of the continued faction shown below can be represented in simplest form as a $\frac{a+\sqrt{b}}{c}$ where a, b, and c are integers. Find $a + b + c$.

$$5 + \cfrac{4}{7 + \cfrac{4}{7 + ...}}$$

ANSWERS

(1 pt) 1. $x =$ ________________

(2 pts) 2. ________________

(3 pts) 3. $a + b + c =$ ________________

Quaboag, Douglass, QSC

All answers must be in simplest exact form.

1. The dimensions of a rectangle are 9cm by 13cm. If each dimension is reduced by x cm, the area is reduced by 57cm^2. Find the new dimensions of the rectangle $\{l, w\}$ (each in cm, either order).

2. One third of Dr. Factor's Algebra I class dropped out during the first term. One half of the remaining students dropped out during the rest of the year. Five sixths of the remaining students failed the course. Two students passed. How many students started Dr. Factor's class at the beginning of first term?

3. Simplify the expression below into a single rational expression of the form $\dfrac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ have no roots in common, the leading coefficient of $q(x)$ is one, and $p(x)$ may be constant. Find $p(1) + q(1)$.

$$\frac{2x}{x^2-4} + \frac{3x}{x^2-3x+2} + \frac{5x+7}{2-x-x^2}$$

ANSWERS

(1 pt) 1. $\{l, w\} = \{$ ______________ $\}$

(2 pts) 2. ______________________

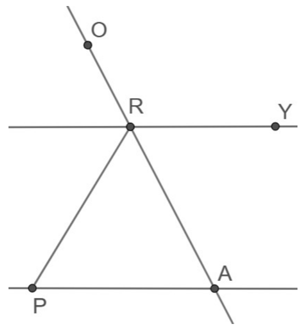
(3 pts) 3. $p(1) + q(1) =$ ____________

Southbridge, Auburn, Algonquin

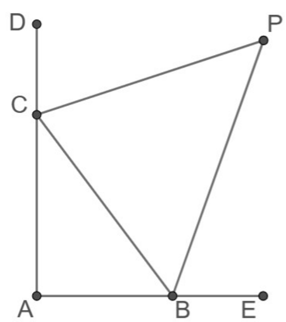
Worcester County Mathematics League
Varsity Meet 2 - December 1, 2021
Round 3 - Parallel Lines and Polygons

All answers must be in simplest exact form.

1. Given $\overleftrightarrow{PA} || \overleftrightarrow{RY}$; $m\angle PRY = 120°$; and $m\angle PAR = 55°$ in the figure below. Find $m\angle PRO$.



2. Given the figure below, where $\overrightarrow{AC} \perp \overrightarrow{AB}$ and the bisectors of $\angle BCD$ and $\angle CBE$ intersect at the point P. Find the measure of $\angle P$.



3. A regular hexagon with side length 2 is divided into four triangles by three diagonals. The largest possible difference between the areas of the largest and the smallest of the four triangles can be expressed as $a\sqrt{b}$. Find the ordered pair (a, b).

ANSWERS

(1 pt) 1. $m\angle PRO =$ ________________ $\circ$

(2 pts) 2. $m\angle P =$ ________________ $\circ$

(3 pts) 3. $(a, b) =$ ________________

Doherty, Shrewsbury, AMSA

Worcester County Mathematics League
Varsity Meet 2 - December 1, 2021
Round 4 - Sequences and Series

All answers must be in simplest exact form.

1. For what values of x will $x+2, 3x+1, 4x-5$, in that order, form an arithmetic sequence?

2. In the sequence shown below that starts with 1, followed by 1, 2, followed by 1, 2, 3, etc., what is the 100^{th} term?

$$1, 1, 2, 1, 2, 3, 1, 2, 3, 4, 1, 2, 3, 4, 5, ...$$

3. The harmonic mean of two numbers x and y is given by $h(x,y) = \dfrac{2}{\frac{1}{x}+\frac{1}{y}} = \dfrac{2xy}{x+y}$. A sequence $a_1, a_2, a_3, ...$ is harmonic if each term other the first term is the harmonic mean of its immediate neighbors, that is, $a_n = h(a_{n-1}, a_{n+1})$. Find the 5^{th} term (a_5) of the harmonic sequence whose first two terms are $a_1 = 15$ and $a_2 = 16$.

ANSWERS

(1 pt) 1. $x =$ ______________________

(2 pts) 2. ______________________

(3 pts) 3. $a_5 =$ ______________________

Doherty, Hudson, QSC

Worcester County Mathematics League
Varsity Meet 2 - December 1, 2021
Round 5 - Matrices and Systems of Equations

All answers must be in simplest exact form.

1. Tony scored 25 points in a recent basketball game. If his score resulted from a combination of two point and three point baskets, with m being the number of 3 point baskets and n the number of two point baskets, and he made 11 baskets in total, find the ordered pair (m, n).

2. Given the two determinants shown below, find the ordered pair (x, y).

$$\begin{vmatrix} x & 5 & -3 \\ 4 & 0 & -1 \\ 2 & 1 & x \end{vmatrix} = -41 \qquad \textit{and} \qquad \begin{vmatrix} 3 & 1 & 5 \\ -1 & y & 0 \\ 1 & 1 & 2 \end{vmatrix} = 1$$

3. Consider a parabola whose equation is of the form $y = Ax^2 + Bx + C$ that passes through the points $(2, 15)$, $(3, 13)$, and $(6, -17)$. Find A, B, and C and express your answer as an ordered triple (A, B, C).

ANSWERS

(1 pt) 1. $(m, n) =$ (________________)

(2 pts) 2. $(x, y) =$ (________________)

(3 pts) 3. $(A, B, C) =$ (________________)

Bancroft, Hudson, Shrewsbury

Round 1 - Fractions, Decimals, and Percents

1. A decrease of 30% followed by an increase of 10% is equivalent to an increase of 40% followed by a decrease of x%. Find x.

Solution: Increasing or decreasing a quantity by x% is equivalent to multiplying the quantity by $\frac{100+x}{100}$ or $\frac{100-x}{100}$. Writing the corresponding ratios from left to right in an equation:

$$\left(\frac{100-30}{100}\right)\left(\frac{100+10}{100}\right)=\left(\frac{100+40}{100}\right)\left(\frac{100-x}{100}\right)$$
$$\left(\frac{70}{100}\right)\left(\frac{110}{100}\right)=\left(\frac{140}{100}\right)\left(\frac{100}{100}-\frac{x}{100}\right)$$

The first three fractions can be simplified by cancelling terms common to the numerator and denominator.

$$\left(\frac{7\cdot 10}{10\cdot 10}\right)\left(\frac{11\cdot 10}{10\cdot 10}\right)=\left(\frac{7\cdot 20}{5\cdot 20}\right)\left(1-\frac{x}{100}\right)$$
$$\left(\frac{7}{10}\right)\left(\frac{11}{10}\right)=\left(\frac{7}{5}\right)\left(1-\frac{x}{100}\right)$$

Solving for x:

$$\left(\frac{5}{7}\right)\left(\frac{7}{10}\right)\left(\frac{11}{10}\right)=1-\frac{x}{100}$$
$$\frac{55}{100}=1-\frac{x}{100}$$
$$\frac{x}{100}=1-\frac{55}{100}=\frac{45}{100}$$
$$x=\boxed{45}$$

2. Convert the repeating decimal $0.2\overline{17}$ to a fraction $\frac{a}{b}$ in lowest terms.

Solution: Let $X=0.2\overline{17}$. Then $1000X=217.\overline{17}$ and $10X=2.\overline{17}$. Subracting:

$$990X=1000X-10X=217.\overline{17}-2.\overline{17}=215$$

Solving for X and simplifying:

$$X=\frac{215}{990}=\frac{5\cdot 43}{5\cdot 198}=\boxed{\frac{43}{198}}$$

3. The value of the continued faction shown below can be represented in simplest form as a $\frac{a+\sqrt{b}}{c}$ where a, b, and c are integers. Find $a+b+c$.

$$5+\cfrac{4}{7+\cfrac{4}{7+...}}$$

Solution: Let the answer be equal to $5+Y$ where:

$$\begin{aligned} Y &= \cfrac{4}{7+\cfrac{4}{7+...}} \\ &= \frac{4}{7+Y} \end{aligned}$$

Multiplying both sides by $7+Y$ and then subtracting 4 results in a quadratic equation:

$$\begin{aligned} Y\left(Y+7\right) &= 4 \\ Y^2+7Y &= 4 \\ Y^2+7Y-4 &= 0 \end{aligned}$$

Applying the quadratic formula:

$$\begin{aligned} Y &= \frac{-7 \pm \sqrt{7^2-4(-4)}}{2} \\ &= \frac{-7 \pm \sqrt{49+16}}{2} \\ &= \frac{-7 \pm \sqrt{65}}{2} \end{aligned}$$

Note that Y is positive, so the extraneous solution $\dfrac{-7-\sqrt{65}}{2}$ is discarded. Then $5+Y = \dfrac{10}{2}+Y = \dfrac{10-7+\sqrt{65}}{2}$, $a=3, b=65, c=2$, and $a+b+c = \boxed{70}$.

Round 2 - Algebra I

1. **(Algebra I)** The dimensions of a rectangle are 9 cm by 13cm. If each dimension is reduced by x cm, the area is reduced by 57 cm^2. Find the new dimensions of the rectangle $\{l, w\}$ (each in cm, either order).

Solution: The original area of the rectangle is $l \cdot w = 117cm^2$, and the new area is equal to $l \cdot w = (9-x)(13-x) = (117-57)\,cm^2$. This equation is converted to a factorable quadratic by applying the distributive law:

$$\begin{aligned}(9-x)(13-x) = x^2 - 22x + 117 &= 117 - 57\\ x^2 - 22x + 57 &= 0\\ x^2 - 19x - 3x + 57 &= 0\\ x(x-19) - 3(x-19) &= 0\\ (x-3)(x-19) &= 0\end{aligned}$$

and $x = 19$ or $x = 3$. The value $x = 19$ is extraneous because it results in negative dimensions, so $x = 3$. Thus, $\{l, w\} = \boxed{\{6, 10\}}$.

2. One third of Dr. Factor's Algebra I class dropped out during the first term. One half of the remaining students dropped out during the rest of the year. Five sixths of the remaining students failed the course. Two students passed. How many students started Dr. Factor's class at the beginning of first term?

Solution: Let x be the number of students in Dr. Factor's class at the start of the first term. Then $\frac{2x}{3}$ students remain after one third drop out, and $\frac{1}{2} \cdot \frac{2x}{3} = \frac{x}{3}$ students remain at the end of the year. Once sixth of those students passed the class since five sixth students failed. Thus:

$$\begin{aligned}\frac{1}{6} \cdot \frac{1}{2} \cdot \frac{2x}{3} &= 2\\ \frac{x}{18} &= 2\end{aligned}$$

and $x = 2 \cdot 18 = \boxed{36}$.

3. Simplify the expression below into a single rational expression of the form $\dfrac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ have no roots in common, the leading coefficient of $q(x)$ is one, and $p(x)$ may be constant. Find $p(1) + q(1)$.

$$\frac{2x}{x^2-4} + \frac{3x}{x^2-3x+2} + \frac{5x+7}{2-x-x^2}$$

Solution: First, factor the denominators of the three rational expressions:

$$\frac{p(x)}{q(x)} = \frac{2x}{(x-2)(x+2)} + \frac{3x}{(x-2)(x-1)} + \frac{5x+7}{(2+x)(1-x)}$$

Next, multiply each of the rational expressions by the missing factor so that they have the common denominator $(x-2)(x+2)(x-1)$, noting that $1-x = (-1)(x-1)$:

$$\frac{p(x)}{q(x)} = \frac{2x(x-1)}{(x-2)(x+2)(x-1)} + \frac{3x(x+2)}{(x-2)(x+2)(x-1)} + \frac{(-1)(x-2)(5x+7)}{(x-2)(x+2)(x-1)}$$

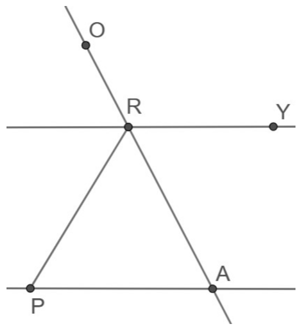
Next, express the sum of rational expressions at a single rational expression, distribute, and combine like terms:

$$\begin{aligned}
\frac{p(x)}{q(x)} &= \frac{2x(x-1)+3x(x+2)-(x-2)(5x+7)}{(x-2)(x+2)(x-1)} \\
&= \frac{2x^2-2x+3x^2+6x-(5x^2-10x+7x-14)}{(x-2)(x+2)(x-1)} \\
&= \frac{2x^2+3x^2-5x^2-2x+6x+10x-7x+14)}{(x-2)(x+2)(x-1)} \\
&= \frac{(2+3-5)x^2(-2+6+10-7)x+14}{(x-2)(x+2)(x-1)} \\
&= \frac{0x^2+7x+14}{(x-2)(x+2)(x-1)} \\
&= \frac{7(x+2)}{(x-2)(x+2)(x-1)} \\
&= \frac{7}{(x-2)(x-1)}
\end{aligned}$$

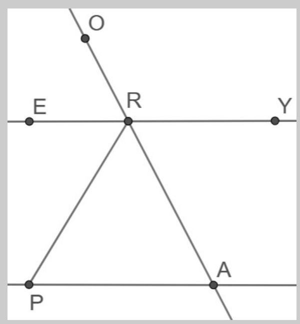
Then $p(1) = 7$, $q(1) = (-1)(0) = 0$ and $p(1) + q(1) = \boxed{7}$.

Round 3 - Parallel Lines and Polygons

1. Given $\overleftrightarrow{PA} || \overleftrightarrow{RY}$; $m\angle PRY = 120°$; and $m\angle PAR = 55°$ in the figure below. Find $m\angle PRO$.



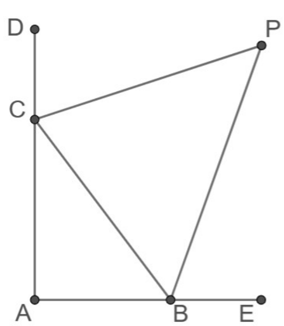
Solution:



First, add the point E to the figure as shown above, and note that $m\angle PRO = m\angle PRE + m\angle ORE$ because they are adjacent angles that form $\angle PRO$. Now $\angle ORE \cong \angle PAR$ because they are corresponding angles, and $\angle PRE$ is supplementary to $\angle PRY$ because they are a linear pair. Therefore $m\angle ORE = m\angle PAR = 55°$, and $m\angle PRE = 180 - m\angle PRY = 180 - 120 = 60°$.

In conclusion, $m\angle PRO = m\angle PRE + m\angle ORE = 60 + 55 = \boxed{115}^\circ$.

2. Given the figure below, where $\overrightarrow{AC} \perp \overrightarrow{AB}$ and the bisectors of $\angle BCD$ and $\angle CBE$ intersect at the point P. Find the measure of $\angle P$.



Solution: From the figure, $m\angle P = 180 - (m\angle PCB + m\angle PBC)$ because the measures of the interior angles of a triangle sum to 180°. Also, $m\angle PCB = \dfrac{m\angle BCD}{2}$ and $m\angle CBP = \dfrac{m\angle CBE}{2}$ because $\overrightarrow{CP}$ bisects $\angle BCD$ and $\overrightarrow{BP}$ bisects $\angle CBE$. Again, from the figure, $m\angle BCD = 180 - m\angle ACB$ and $m\angle CBE = 180 - m\angle ABC$. Putting these results together:

$$\begin{aligned} m\angle PCB + m\angle PBC &= \frac{m\angle BCD + m\angle CBE}{2} \\ &= \frac{180 - m\angle ACB + 180 - m\angle ABC}{2} \\ &= \frac{360 - (m\angle ACB + m\angle ABC)}{2} \end{aligned}$$

Note that $\angle ACB$ and $\angle ABC$ are the acute angles of right triangle ΔABC so that $m\angle ACB + m\angle ABC = 90°$. Thus $m\angle PCB + m\angle PBC = (360 - 90)/2 = 135°$, and $m\angle P = 180 - 135 = \boxed{45}^\circ$.

3. A regular hexagon with side length 2 is divided into four triangles by three diagonals. The largest possible difference between the areas of the largest and the smallest of the four triangles can be expressed as $a\sqrt{b}$. Find the ordered pair (a, b).

Solution: The triangle with the smallest possible area is a triangle whose three vertices are consecutive vertices of the hexagon, for instance, ΔABC of regular hexagon $ABCDEF$. The triangle with the largest possible area is created when the other three triangles are congruent with the smallest possible area. In regular hexagon $ABCDEF$, this situation occurs when the three diagonals are $\overline{AC}, \overline{CE}$, and $\overline{AE}$, creating small triangles ΔABC, ΔCDE, and ΔEFA, as well as large triangle ΔACE.

Each small triangle is an isosceles triangle with legs of length 2 (they are sides of the hexagon) and whose vertex angle is an interior angle of the hexagon and therefore measures 120°. The altitude from the vertex angle bisects the vertex angle and divides the isosceles triangle into congruent $30-60-90$ triangles each with hypotenuse of length 2 (the side length of the hexagon) and legs of length 1 and $\sqrt{3}$. The area of a small triangle is $\frac{1}{2}bh$, where $h = 1$ is the height (the length of the short leg of the 30-60-90 triangle) and $b = 2\sqrt{3}$ is the base (twice the length of the longer leg of the 30-60-90 triangle). Thus, the area of the smallest triangle is $\dfrac{2\sqrt{3} \cdot 1}{2} = \sqrt{3}$.

The largest possible triangle ΔACE is equilateral. Each of its sides is a diagonal and also the longest side of one of the three congruent small triangles. Therefore the side length of ΔACE is $s = b = 2\sqrt{3}$. The area of an equilateral triangle with side length s is $\dfrac{s^2\sqrt{3}}{4}$. So, the area of the large triangle is $\dfrac{\left(2\sqrt{3}\right)^2\sqrt{3}}{4} = \dfrac{12\sqrt{3}}{4} = 3\sqrt{3}$. Finally, the difference between the the largest and smallest areas is $3\sqrt{3} - \sqrt{3} = 2\sqrt{3} = a\sqrt{b}$, $a = 2$ and $b = 3$, so $(a, b) = \boxed{(2, 3)}$.

Round 4 - Sequences and Series

1. For what values of x will $x+2, 3x+1, 4x-5$, in that order, form an arithmetic sequence?

Solution: Note that, given three consecutive terms of an arithmetic sequence, the middle term is always equal to the arithmetic mean of its two adjacent terms. Setting up this equation and solving for x:

$$\begin{aligned}\frac{(x+2)+(4x-5)}{2} &= 3x+1\\ x+2+4x-5 &= 2(3x+1)\\ 5x-3 &= 6x+2\end{aligned}$$

Subracting $5x+2$ from each side results in $x = \boxed{-5}$.

The answer $x=-5$ is checked by plugging it into the original sequence. The result is $-3, -14, -25$, an arithmetic sequence with common difference -11.

2. In the sequence shown below that starts with 1, followed by 1, 2, followed by 1, 2, 3, etc., what is the 100^{th} term?

$$1, 1, 2, 1, 2, 3, 1, 2, 3, 4, 1, 2, 3, 4, 5, ...$$

Solution: This sequence is a sequence of subsequences that count from 1. The n^{th} subsequence terminates when its count reaches n, which is the m^{th} term of the overall sequence, where $m = \frac{n(n+1)}{2}$. That is, the counting restarts from 1 each time the sequence reaches a number of terms that is a *triangle number*, that is: $m = 1, 3, 6, 10, 15, 21, ...$ Thus, the value of the 100^{th} term can be found by counting up from the largest triangle number less than 100.

This triangle number is found by setting up and solving the inequality for integer n:

$$\begin{aligned}\frac{n(n+1)}{2} &< 100\\ n(n+1) &< 200\\ n^2+n &< 200\end{aligned}$$

This inequality is most quickly solved by guess and check. The largest perfect square less than 200 is $196 = 14^2$. However, $14^2 + 14 = 196 + 14 > 200$, so the solution to the inequality is $n = 13$. The last term in the 13^{th} subsequence is the m^{th} term where $m = \frac{13(13+1)}{2} = 13 \cdot 7 = 91$. Thus, the 14^{th} subsequence counts up to $100 - 91 = \boxed{9}$ to reach the 100^{th} term of the overall sequence.

3. The harmonic mean of two numbers x and y is given by $h(x, y) = \dfrac{2}{\frac{1}{x} + \frac{1}{y}} = \dfrac{2xy}{x+y}$. A sequence $a_1, a_2, a_3, ...$ is harmonic if each term other the first term is the harmonic mean of its immediate neighbors, that is, $a_n = h(a_{n-1}, a_{n+1})$. Find the 5^{th} term (a_5) of the harmonic sequence whose first two terms are $a_1 = 15$ and $a_2 = 16$.

Solution: The fifth term can be found iteratively: find a_3 from a_1 and a_2, then find a_4 from a_3 and a_2, and finally find a_5 from a_3 and a_4. First, find a_3:

$$a_2 = \frac{2}{\frac{1}{a_1} + \frac{1}{a_3}}$$

$$16 = \frac{2}{\frac{1}{15} + \frac{1}{a_3}}$$

$$16\left(\frac{1}{15} + \frac{1}{a_3}\right) = 2$$

$$\frac{16}{15} + \frac{16}{a_3} = 2$$

$$\frac{16}{a_3} = 2 - \frac{16}{15} = \frac{14}{15}$$

$$a_3 = \frac{16 * 15}{14} = \frac{120}{7}$$

Now, use a_2 and a_3 to find a_4:

$$\frac{120}{7} = \frac{2}{\frac{1}{16} + \frac{1}{a_4}}$$

$$\frac{120}{7}\left(\frac{1}{16} + \frac{1}{a_4}\right) = 2$$

$$\frac{120}{7 \cdot 16} + \frac{120}{7a_4} = 2$$

$$\frac{120}{7a_4} = 2 - \frac{15}{14} = \frac{13}{14}$$

$$a_4 = \frac{120 \cdot 14}{7 \cdot 13} = \frac{240}{13}$$

Continue similarly to find a_5:

$$\begin{aligned}
\frac{240}{13} &= \frac{2}{\frac{7}{120} + \frac{1}{a_5}} \\
\frac{240}{13}\left(\frac{7}{120} + \frac{1}{a_5}\right) &= 2 \\
\frac{240 \cdot 7}{13 \cdot 120} + \frac{240}{13a_5} &= 2 \\
\frac{240}{13a_5} &= 2 - \frac{14}{13} = \frac{12}{13} \\
a_5 &= \frac{240 \cdot 13}{12 \cdot 13} = \boxed{20}
\end{aligned}$$

Round 5 - Matrices and Systems of Equations

1. Tony scored 25 points in a recent basketball game. If his score resulted from a combination of two point and three point baskets, with m being the number of 3 point baskets and n the number of two point baskets, and he made 11 baskets in total, find the ordered pair (m, n).

Solution: There are two equations in the two unknowns m and n:

$$\begin{aligned} m + n &= 11 \\ 3m + 2n &= 25 \end{aligned}$$

Multiply the first equation by 2 and subtract from the second equation to find $m = 3$. Substitute 3 for m in the first equation to find that $n = 11 - 3$ or $n = 8$. The ordered pair $(m, n) = \boxed{(3, 8)}$.

2. Given the two determinants shown below, find the ordered pair (x, y).

$$\begin{vmatrix} x & 5 & -3 \\ 4 & 0 & -1 \\ 2 & 1 & x \end{vmatrix} = -41 \qquad \textit{and} \qquad \begin{vmatrix} 3 & 1 & 5 \\ -1 & y & 0 \\ 1 & 1 & 2 \end{vmatrix} = 1$$

Solution: Calculating the two determinants results in two equations in unknowns x and y. Applying cofactor expansion on the second row of first determinant:

$$\begin{vmatrix} x & 5 & -3 \\ 4 & 0 & -1 \\ 2 & 1 & x \end{vmatrix} = -4 \begin{vmatrix} 5 & -3 \\ 1 & x \end{vmatrix} + 0 \begin{vmatrix} x & -3 \\ 2 & x \end{vmatrix} - (-1) \begin{vmatrix} x & 5 \\ 2 & 1 \end{vmatrix}$$

$$= -4\,(5x - (-3)) + 0 + x - 10 = -20x - 12 + x - 10 = -19x - 22 = -41$$

And $19x = 41 - 22 = 19$, so $x = 1$. Calculating the second determinant by expanding cofactors on its second row:

$$\begin{vmatrix} 3 & 1 & 5 \\ -1 & y & 0 \\ 1 & 1 & 2 \end{vmatrix} = -(-1) \begin{vmatrix} 1 & 5 \\ 1 & 2 \end{vmatrix} + y \begin{vmatrix} 3 & 5 \\ 1 & 2 \end{vmatrix} - 0 \begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix}$$

$$= 1(2 - 5) + y(6 - 5) + 0 = -3 + y = 1$$

Therefore $y = 1 + 3 = 4$ and $(x, y) = \boxed{1, 4}$.

3. Consider a parabola whose equation is of the form $y = Ax^2 + Bx + C$ that passes through the points $(2, 15)$, $(3, 13)$, and $(6, -17)$. Find A, B, and C and express your answer as an ordered triple (A, B, C).

Solution: Create a system of three equations in the three unknowns A, B and C by substituting each of the x and y values of the three points, in turn, into the equation $y = Ax^2 + Bx + C$:

$$15 = A2^2 + B2 + C = 4A + 2B + C$$
$$13 = A3^2 + B3 + C = 9A + 3B + C$$
$$-17 = A6^2 + B6 + C = 36A + 6B + C$$

Subtract the first equation from the second equation and then subtract the second equation from the third equation to eliminate C and create a system of two equations in unknowns A and B:

$$-2 = 5A + B$$
$$-30 = 27A + 3B$$

Next, subtract three times the first equation above from the second equation to eliminate B:

$$-30 - 3(-2) = 27A - 3(5A) + 3B - 3B$$
$$-30 + 6 = 27A - 15A$$
$$-24 = 12A$$

and $A = -2$. Next, substitute this value in the first equation of the system of two equations to find B:

$$-2 = 5(-2) + B$$
$$-2 + 10 = 8 = B$$

Finally, substitute $A = -2$ and $B = 8$ into the first of the system of three equations to find C:

$$15 = 4(-2) + 2(8) + C$$
$$15 + 8 - 16 = C$$

and $C = 7$. The final solution is $(A, B, C) = \boxed{(-2, 8, 7)}$, in that order.

Team Round

1. The membership ratio of adults to minors at the Racquet Club is 3 : 2. A membership drive was held and 150 minors joined the club. The new ratio of adults to minors is 2 : 3. Find the new total membership of the Raquet Club.

Solution: Let m be the original number of minors and a be the number of adults members. Then $m : a = 2x : 3x$ and $a : (m + 150) = 2y : 3y$ for some integers x and y. Let $a = 6z$, where $6z = 3x = 2y$ so that $m : a : (m + 150) = 4z : 6z : 9z$ and $9z - 4z = 5z = m + 150 - m = 150$. Therefore $5z = 150$ and $z = 30$. There are $6z$ adults and $9z$ minors in the new membership, for a total of $(6 + 9)z = 15(30) = \boxed{450}$ members.

2. Mr. Ford traveled at 22 miles per hour (mph) for t hours and 27 mph for h hours, traveling a combined distance of 282 miles. If during each segment his speed was 5 mph faster, he would have traveled a total of 337 miles. Find the ordered pair, (t, h).

Solution: The condition that Mr. Ford traveled 282 miles is respresented by the equation $22t + 27h = 282$, and the second condition is represented by the equation $(22 + 5)t + (27 + 5)h = 337$. Subtracting the first equation from the second results in $5t + 5h = 337 - 282 = 55$, or $t + h = 11$, or $22t + 22h = 11 \cdot 22 = 242$. Subtracting this new equation from the first equation results in $5h = 282 - 242 = 40$, or $h = 8$. Since $t + h = 11$, $t = 3$ and $(t, h) = \boxed{(3, 8)}$.

3. If

$$\begin{vmatrix} 3 & 0 & x \\ -1 & 4 & 5 \\ 2 & x & x \end{vmatrix} = \begin{vmatrix} 5 & x & x \\ x & 5 & x \\ 2 & x & 0 \end{vmatrix}$$

Find all possible values for x.

Solution: Using cofactor expansion and expanding the left determinant on the first row yields $3(4x - 5x) + x(-x - 8) = -3x - 8x - x^2 = -x(11 + x)$. Expanding the right determinant on the third row yields $2(x^2 - 5x) - x(5x - x^2) = 2x^2 - 10x - 5x^2 + x^3 = x^3 + (2-5)x^2 - 10x = x^3 - 3x^2 - 10x = x(x^2 - 3x - 10)$. Setting the two determinants equal results in $-x(11 + x) = x(x^2 - 3x - 10)$, and adding $x(11 + x)$ to both sides results in $x(x^2 - 3x - 10) + x(x + 11) = x(x^2 - 3x + x - 10 + 11) = x(x^2 - 2x + 1) = 0$. Factoring the perfect square quadratic yields $x(x - 1)^2 = 0$, and $x \in \boxed{\{0, 1\}}$.

4. Jean has p pencils and j jars. If she puts 4 pencils in each jar,she will have 1 jar left over. If she puts 3 pencils in each jar, she will have 1 pencil left over. Find the ordered pair (p, j).

Solution: The first condition specifies that the number of pencils is equal to four times one less than the number of jars, or $p = 4(j-1) = 4j-4$. The second condition specifies that the number of pencils is equal to three times the number of jars plus one, the left over pencil, or $p = 3j + 1$. Subtracting these equations yields $j - 5 = 0$, so $j = 5$. Substituting $j = 5$ into either equation results in $p = 16$, so $(p, j) = \boxed{(16, 5)}$.

5. The exterior angles at each of the vertices of a pentagon are in the ratio $1 : 2 : 2 : 6 : 7$. What is the degree measure of the smallest <u>interior</u> angle?

Solution: The sum of the exterior angles of any polygon is 360 degrees. Thus, from the ratio $x : 2x : 2x : 6x : 7x$, $x + 2x + 2x + 6x + 7x = 360$, so $18x = 360$ and $x = 20$. The smallest interior angle corresponds with the largest exterior angle, the $7x$ angle, which measures $140°$. The smallest interior angle is therefore $180 - 140 = \boxed{40}^\circ$.

6. If

$$\sum_{b=2}^{4}(a^2b - ab) = \sum_{c=3}^{5}(ac + 6)$$

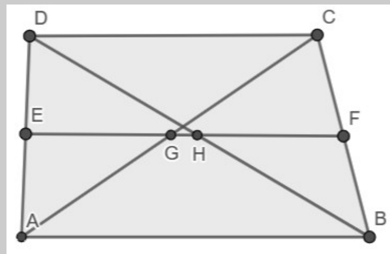
find all values of a.

Solution: The first expression simplifes to $(a^2 - a)\sum_{b=2}^{4} b = (a^2 - a)(2 + 3 + 4) = (a^2 - a) \cdot 9$. The second expression simplifies to $(3a+6)+(4a+6)+(5a+6) = 12a+18$. Thus we have $9a^2-9a = 12a+18$ so that $9a^2 - 21a - 18 = 0$, or $3a^2 - 7a - 6 = 0$. This quadratic can be factored by grouping. Note that $3\cdot(-6) = -18$, that $-18 = -9\cdot 2$, and that $-9+2 = -7$, so $3a^2 - 7a - 6 = 3a^2 - 9a + 2a - 6 = 0$.

Proceeding with the factoring: $3a(a - 3) + 2(a - 3) = (3a + 2)(a - 3) = 0$, and $a \in \boxed{\left\{-\frac{2}{3}, 3\right\}}$.

7. $ABCD$ is a trapezoid with $\overline{AB} \parallel \overline{DC}$. Let the midpoint of $\overline{AD}$ be E, the midpoint of $\overline{BC}$ be F, and let median $\overline{EF}$ intersect diagonals $\overline{AC}$ and $\overline{BD}$ at G and H respectively. If $AB = 30$ cm, $CD = 25$ cm, how long is $\overline{GH}$?

Solution:



The trapezoid is drawn above with the vertices and midpoints labeled. Median $\overline{EG}$ is parallel to $\overline{AB}$ and $\overline{CD}$, in particular. Also, $EF = (AB + CD)/2 = 27.5$. Note that $\overline{FH}$ bisects side $\overline{BD}$ of ΔDBC and $\overline{EG}$ bisects side $\overline{AC}$ of ΔDAC because *a line parallel to one side of a triangle that bisects a second side also bisects the third side.* Therefore G and H are midpoints of $\overline{AC}$ and $\overline{BD}$ and $FH = CD/2 = EG$ because *a line segment that joins the midpoints of two sides of a triangle is half the length of the third side.* Thus, $EG + FH = CD = 25$. Finally, $GH + EG + HF = EF$, or $GH + 25 = 27.5$, and $GH = \boxed{2.5}$.

8. Bill has 10 L of a 65% salt solution. After water evaporates from the solution, an 85% salt solution remains. Find the volume of the remaining water. Answer as a fraction in lowest terms.

Solution: The amount of salt before and after the evaporation is equal. Thus, $.65 \cdot 10\text{L} = .85 \cdot x$ L. Writing the means/extremes of this proportion (taking the cross product): $.65 * 10 = .85 * x$. Thus $x = 6.5/.85 = 650/85 = \boxed{130/17}$.

9. If t_n denotes the n^{th} term of a sequence, $t_{n+1} = 2t_n - 5$ and $t_6 = t_3 + 35$, find t_2.

Solution: Using the recursive relation, $t_6 = 2t_5 - 5 = 2(2t_4 - 5) - 5 = 2(2(2t_3 - 5) - 5) - 5 = 8t_3 - 35$. This is also equal to $t_3 + 35$, so $8t_3 - 35 = t_3 + 35, 7t_3 = 70, t_3 = 10$. Using the recursive relation, $10 = t_3 = 2t_2 - 5$, so $2t_2 = 15$ and $t_2 = \boxed{7.5}$.

Worcester County Mathematics League

Varsity Meet 2 - December 1, 2021

Answer Key

Round 1 - Fractions, Decimals, and Percents

1. 45
2. $\frac{43}{198}$
3. 70

Round 2 - Algebra I

1. $\{6, 10\}$ (either order)
2. 36
3. 7

Round 3 - Parallel Lines and Polygons

1. 115 or 115° or 115 degrees
2. 45 or 45° or 45 degrees
3. $(2, 3)$ (in that order)

Round 4 - Sequences and Series

1. -5
2. 9
3. 20

Round 5 - Matrices and Systems of Equations

1. $(3, 8)$ (in that order)
2. $(1, 4)$ (in that order)
3. $(-2, 8, 7)$ (in that order)

Team Round

1. 450
2. $(3, 8)$ (in that order)
3. $\{0, 1\}$ (either order) or $0, 1$
4. $(16, 5)$ (in that order)
5. 40 or 40° or 40 degrees
6. $\{-\frac{2}{3}, 3\}$ (either order) or $-\frac{2}{3}, 3$
7. $\frac{5}{2}$ or 2.5 or $2\frac{1}{2}$
8. $\frac{130}{17}$
9. $\frac{15}{2}$ or 7.5 or $7\frac{1}{2}$

Worcester County Mathematics League
Varsity Meet 2 - December 1, 2021
Team Round

All answers must be in simplest exact form.

1. The membership ratio of adults to minors at the Racquet Club is 3 : 2. A membership drive was held and 150 minors joined the club. The new ratio of adults to minors is 2 : 3. Find the new total membership of the Raquet Club.

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Find all possible values for x.

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Worcester Academy, Clinton, Worcester Academy, Grafton, NDA, Shrewsbury, Shepherd Hill, Tantasqua, Auburn

Worcester County Mathematics League
Varsity Meet 2 - December 1, 2021
Team Round Answer Sheet

ANSWERS

1. ______________________________

2. ______________________________

3. ______________________________

4. ______________________________

5. ______________________________

6. ______________________________

7. ______________________________

8. ______________________________

9. ______________________________